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SCHUR-OSTROWSKI THEOREMS FOR FUNCTIONALS ON $L_1(0,1)$

by

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that certain functions are increasing with respect to this ordering. Such functions are
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has been extended to elements of $L_1(0,1)$ by Ryff (1963). A functional on $L_1(0,1)$ that is
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ITEM #19, ABSTRACT, CONTINUED: $L_1(0,1)$ with respect to the partial ordering of unrestricted majorization through another analogue of the Schur-Ostrowski condition.

Schur-Ostrowski Theorems for Functionals on $L_1(0,1)$

by

Wai Chan, Frank Proschan, and Jayaram Sethuraman

A-1

ABSTRACT

Hardy, Littlewood and Pólya (1934) introduced the partial
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majorization is said to be Schur-convex. In this paper, we prove
an analogue of the Schur-Ostrowski condition which characterizes
Schur-convex functionals in terms of their Gâteaux differentials.
We also introduce another partial ordering in $L_1(0,1)$ called
unrestricted majorization. This partial ordering is similar to
majorization but does not involve the use of decreasing rearrange-
ments. We establish a characterization of non-decreasing functionals
on $L_1(0,1)$ with respect to the partial ordering of unrestricted
majorization through another analogue of the Schur-Ostrowski condition.

1. Introduction.

Hardy, Littlewood and Polya (1934) introduced the following partial order in n-dimensional Euclidean spaces: a n-vector $\underset{\sim}{x} = (x_1, \ldots x_n)$ majorizes $\underset{\sim}{y} = (y_1, \ldots, y_n)$, $(x \overset{m}{\geq} y$ in symbols), whenever

$$\sum_1^k x_i^* \geq \sum_1^k y_i^*, \quad k = 1, \ldots, n-1$$

and

$$\sum_1^n x_i = \sum_1^n y_i,$$

where $\underset{\sim}{x}^*$, $\underset{\sim}{y}^*$ are the vectors obtained from $\underset{\sim}{x}$ and $\underset{\sim}{y}$ by rearranging their components in decreasing order.

This partial order has been extended to elements of $L_1(0,1)$ by Ryff (1963) and is given in Definition 1.2 below. Before giving this definition, we develop some notation to be used in defining a decreasing rearrangement of a function. Let x be a measurable, real valued function on (0,1) and m be the Lebesgue measure. For each x, one can associate a function d_x on $(-\infty, \infty)$ defined by

$$d_x(s) = m(\{t: x(t) > s\}), \quad -\infty < s < \infty.$$

This function d_x, called the distribution function of x, is non-increasing and right continuous. Two functions x and y are said to be equivalent in distribution if $d_x = d_y$. The right continuous inverse of d_x, denoted x^*, is defined by

$$x^*(t) = \inf \{s: d_x(s) \leq t\}.$$

The function x^*, which is non-increasing, right continuous and has the same distribution function as x, is called the decreasing rearrangement of x. The functions x and x^* are simultaneously integrable (or non-integrable), and their integrals are related by

$$\int_0^s x^*(t)dt \geq \int_0^s x(t)dt \qquad 0 \leq s < 1,$$

and

$$\int_0^1 x^*(t)dt = \int_0^1 x(t)dt.$$

The following theorem due to Ryff (1970) shows that by composing the decreasing rearrangement of a function with a measure preserving transformation, one can recover the original function.

1.1. Theorem. To each $x \in L_1(0,1)$, there corresponds a measure preserving transformation $\sigma: (0,1) \to (0,1)$ such that $x(t) = x^*[\sigma(t)]$, where σ is defined by

$$\sigma(s) = m\{t: x(t) > x(s)\} + m\{t \leq s: x(t) = x(s)\} \ .$$

The definition of the partial ordering of majorization of elements in $L_1(0,1)$, due to Ryff (1963), is given below.

1.2. Definition. Let $x, y \in L_1(0,1)$. We say that x majorizes y, $(x \overset{m}{\geq} y$ in symbols) if

$$\int_0^s x^*(t)dt \geq \int_0^s y^*(t)dt, \quad 0 \leq s < 1,$$

and

$$\int_0^1 x(t)dt = \int_0^1 y(t)dt,$$

where x^* and y^* are the decreasing rearrangements of x and y, respectively.

Several authors [see, eg. Day (1973), Chong (1976)] have obtained interesting results using this partial ordering. It is also related to the variability ordering of Ross (1982).

By removing the rearrangement requirement in Definition 1.2, we obtain a different ordering called unrestricted majorization, as defined below.

1.3. Definition. Let x, y ∈ $L_1(0,1)$. We say that x dominates y in
the ordering of unrestricted majorization, (x $\overset{U}{\geq}$ y in symbols), if

$$\int_0^s x(t)dt \geq \int_0^s y(t)dt, \quad 0 \leq s < 1,$$

and

$$\int_0^1 x(t)dt = \int_0^1 y(t)dt.$$

The ordering of unrestricted majorization as applied to the
class of density functions leads to the usual stochastic ordering
as seen below:

Let X and Y be random variables on (0,1) with densities f and
g respectively. If f $\overset{U}{\geq}$ g, then $\int_0^s f \geq \int_0^s g$ for all 0 < s < 1, or
$P(X \leq s) \geq P(Y \geq s)$. Thus the condition X $\leq^{st}$ Y is equivalent to
f $\overset{U}{\geq}$ g.

Many inequalities that arise from majorization in the finite dimen-
sional case can be extended for elements of $L_1(0,1)$. Ryff (1967)
proved the following analogue of Muirhead's inequality.

1.4. Theorem. Let x and y be bounded measurable functions on (0,1).
If x $\overset{m}{\geq}$ y and u is a positive function such that u ∈ $L_1(0,1)$ for all
p, $-\infty < p < \infty$, then $\int_0^1 \log[\int_0^1 u(t)^{x(s)}dt]ds \geq \int_0^1 \log[\int_0^1 u(t)^{y(s)}dt]ds$.
Conversely, if the inequality holds for all such u, then x $\overset{m}{\geq}$ y.

In the discrete case, Muirhead's inequality can be reformulated
by identifying an appropriate function which preserves the ordering
of majorization. Such functions are said to be Schur-convex. Schur
(1923) and Ostrowski (1952) gave necessary and sufficient conditions

for a function to be Schur-convex in terms of their partial derivatives. We quote from Marshall and Olkin (1979) about the importance of this result, "it is difficult to overemphasize the usefulness of the (Schur-Ostrowski) condition, ..., many or even most of the theorems giving Schur-convexity were first discovered by checking (the Schur-Ostrowski condition)." In the next section, we will present an analogue of this result for Schur-convex functionals on $L_\infty(0,1)$. This result, given in Theorem 2.9, is then used to characterize Schur-convex functionals on $L_1(0,1)$. We also characterize non-decreasing functionals on $L_1(0,1)$ with respect to the partial ordering of unrestricted majorization through another analogue of the Schur-Ostrowski condition. These results will be used to prove the generalized Muirhead's Theorem (Proschan and Sethuraman, 1976) in Section 3. An application to peakedness comparisons of distributions is discussed in Section 3.

2. Main Theorems.

We first proceed with some definitions.

2.1. <u>Definition.</u> A functional ϕ defined on a set $A \subseteq L_1(0,1)$ is said to be Schur-convex on A if y_1, $y_2 \in A$ and $y_1 \overset{m}{\geq} y_2$ imply that $\phi(y_1) \geq \phi(y_2)$.

A Schur-convex functional is necessarily constant over functions that are equivalent in distribution. Thus for a Schur-convex functional ϕ, the value $\phi(x)$ depends only on the distribution function of x. A set A is said to be invariant if $x \in A$ and x and y are equivalent in distribution imply that $y \in A$. Henceforth, we shall only consider Schur-convex functionals on an invariant set.

For a characterization of Schur-convex functionals, we need the following notion of directional derivative.

2.2. Definition. Let ϕ be functional defined on a convex set $A \subseteq L_1(0,1)$. Let $y \in A$ and h be such that $y + \theta h \in A$ for all sufficiently small θ. The Gâteaux differential of ϕ at y in the direction of h is defined to be

$$\frac{\partial \phi}{\partial h}(y) = \lim_{\theta \to 0} \frac{\phi(y+\theta h) - \phi(y)}{\theta}$$

if the limit exists.

Note that $\frac{\partial \phi}{\partial h}(y)$ is simply the derivative, at $\theta = 0$, of the real valued function on $[0,1]$ defined by $\psi(\theta) = \phi(y+\theta h)$.

Let $\mathcal{D}_1$ be the class of decreasing functions in $L_1(0,1)$, let $\mathcal{D}_\infty$ be the class of decreasing functions in $L_\infty(0,1)$. Let

$T = \{h: h = \lambda_1 I_{(a,b)} + \lambda_2 I_{(c,d)}$, where $0 \le a < b < c < d \le 1$, $\lambda_1 \ge 0 \ge \lambda_2$,

$\lambda_1(b-a) + \lambda_2(d-c) = 0\}$. The class T consists of step functions h which take at most two non-zero values, are decreasing on its support and satisfy $\int_0^1 h(t)dt = 0$. Note that $h \in T$ implies $h \overset{m}{\ge} 0$.

Let $y \in \mathcal{D}_1$ and $h \in T$. Then $y + h$ need not be decreasing. However, we have $y + h \overset{m}{\ge} y$, as given in the next lemma.

2.3 Lemma. Let $y \in \mathcal{D}_1$ and $h \in T$, then $y + h \overset{m}{\ge} y$.

Proof.

Note that

$$\int_0^s (y+h)^* \ge \int_0^s (y+h)$$

$$= \int_0^s y^* + \int_0^s h$$

$$\ge \int_0^s y^*, \quad 0 \le s < 1$$

and

$$\int_0^1 (y+h) = \int_0^1 y + \int_0^1 h = \int_0^1 y.$$

Hence, $y + h \overset{m}{\geq} y$. ‖

In the following Theorem, we give a necessary condition for functionals increasing in the ordering of unrestricted majorization.

2.4. Theorem. Let A be an open subset of $L_1(0,1)$. Let ϕ be a functional defined on A such that ϕ is non-decreasing with respect to the ordering of unrestricted majorization. Let $y \in A$ and $h \in T$. Suppose that the Gâteaux differential $\frac{\partial \phi}{\partial h}(y)$ exists. Then $\frac{\partial \phi}{\partial h}(y) \geq 0$.

Proof.

Since A is open, $y + \theta h \in A$ for all sufficiently small θ. Thus for all sufficiently small positive θ, $y + \theta h$ and y are elements of A and $y + \theta h \overset{u}{\geq} y$. This implies that

$$\phi(y+\theta h) \geq \phi(y)$$

and

$$\frac{\partial \phi}{\partial h}(y) = \lim_{\theta \downarrow 0} \frac{1}{\theta}[\phi(y+\theta h) - \phi(y)]$$

$$\geq 0. \ \|$$

Next, we consider Schur-convex functionals defined on an invariant set A.

2.5. Theorem. Let A be an open invariant subset of $L_1(0,1)$. Let ϕ be a Schur-convex functional defined on A. Let $y \in D_\infty \cap A$ and $h \in T$. Suppose that the Gâteaux differential $\frac{\partial \phi}{\partial h}(y)$ exists. Then $\frac{\partial \phi}{\partial h}(y) \geq 0$.

Proof.

Since A is open, $y + \theta h \in A$ for all sufficiently small θ. Furthermore, for sufficiently small positive θ, $y + \theta h \overset{m}{\geq} y$ from Lemma 2.3. Hence $\phi(y+\theta h) \geq \phi(y)$ and

$$\frac{\partial \phi}{\partial h}(y) = \lim_{\theta \to 0} \frac{1}{\theta}[\phi(y+\theta h) - \phi(y)]$$

$$\geq 0. \;\|$$

To show that this condition is also sufficient, which is the content of the main theorem, Theorem 2.10 of this section, we need the following lemmas.

2.6. Lemma. Let A be a convex subset of $L_1(0,1)$. Let ϕ be a functional defined on an open set containing A. Let $\frac{\partial \phi}{\partial h}(y) \geq 0$ for $y \in A$ and $h \in T$. Then y_1, $y_2 \in A$ and $y_2 - y_1 \in T$ imply that $\phi(y_2) \geq \phi(y_1)$.

Proof.

Let $h = y_2 - y_1 \in T$. For $\theta \in [0,1]$, define

$$y_\theta = y_1 + \theta(y_2 - y_1) = \theta y_2 + (1-\theta)y_1,$$

and

$$\psi(\theta) = \phi(y_\theta).$$

Note that y_θ is in A. Now,

$$\frac{d}{d\theta}\psi(\theta) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\psi(\theta+\epsilon) - \psi(\theta)]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\phi(y_\theta+\epsilon h) - \phi(y_\theta)]$$

$$= \frac{\partial \phi}{\partial h}(y_\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq 1.$$

Hence,

$$\phi(y_2) - \phi(y_1) = \int_0^1 \frac{d}{d\theta}\psi(\theta)d\theta$$

$$= \int_0^1 \frac{\partial\phi}{\partial h}(y_\theta)d\theta \geq 0. \; \|$$

In the next lemma, we show that if y_1, y_2 are step functions such that $y_2 \overset{y}{\geq} y_1$, then $y_2 - y_1$ can be written as the sum of functions in T.

2.7. Lemma. Let y_1, y_2 be step functions on $(0,1)$ such that $y_2 \overset{y}{\geq} y_1$. Then there exist $h_1, \ldots, h_N$ in T such that

$$(2.1) \qquad\qquad y_2 = y_1 + \sum_{i=1}^{N} h_i.$$

Proof. There is nothing to prove if $y_1 = y_2$.

Let $y_1 \neq y_2$. Since y_1 and y_2 are step functions, there is an integer $n \geq 2$, such that

$$y_2(t) - y_1(t) = \sum_{i=1}^{n} a_i I_{(c_i,d_i)}(t) \quad \text{where} \quad a_i \neq 0,$$

(c_i,d_i) are disjoint intervals and $0 \leq c_1 < d_1 < \ldots < c_n < d_n \leq 1$. Note that $y_2 \overset{y}{\geq} y_1$ implies that $a_1 > 0$ and $a_n < 0$. We will prove that (2.1) holds with $N \leq n-1$, by an induction on n.

Note that the lemma is immediate when $n = 2$. Assume that the lemma is true for $n = 2, \ldots, k-1$. We will now prove that the lemma

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holds for n = k. Let a_j be the first negative term such that either $a_{j+1} > 0$ or $j = k$. Define a function in T by

$$h = a_1 I_{(c_1,c_1')} + a_j I_{(d_j',d_j)},$$

where $c_1' \leq d_1$ and $d_j' \geq c_j$ are chosen so that $a_1(c_1'-c_1) + a_j(d_j-d_j') = 0$ and one of the following holds:

1) $c_1' = d_1$ and $d_j' = c_j$ if $a_1(d_1-c_1) + a_j(d_j-c_j) = 0$,

2) $c_1' < d_1$ and $d_j' = c_j$ if $a_1(d_1-c_1) + a_j(d_j-c_j) > 0$,

3) $c_1' = d_1$ and $d_j' > c_j$ if $a_1(d_1-c_1) + a_j(d_j-c_j) < 0$.

We will now establish that $y_2 \overset{\cup}{\geq} y_1 + h$ by showing that $\int_0^s (y_2-y_1-h) \geq 0$ for all $0 < s \leq 1$. Note that $h = 0$ on the interval $(d_j,1)$.

Let $s > d_j$, then $\int_0^s h = \int_0^1 h = 0$. Thus,

$$\int_0^s (y_2-y_1-h) = \int_0^s (y_2-y_1) \geq 0.$$

Let $0 < s \leq d_j$. Then either $y_2(t) - y_1(t) \geq 0$ for all $0 < t < s$ or $y_2(t) - y_1(t) \leq 0$ for all $s < t \leq d_j$, since there is only one sign change among $a_1, \ldots, a_j$ and the sign changes from positive to negative. Note that h agrees with $y_2 - y_1$ on the intervals (c_1,c_1') and (d_j',d_j), and that h is identically zero outside these intervals.

If $y_2(t) - y_1(t) \geq 0$ for all $0 < t < s$, then $y_2 - y_1 \geq h \geq 0$ on the interval $(0,s)$. This implies that

$$\int_0^s (y_2-y_1-h) \geq 0.$$

If $y_2(t) - y_1(t) \leq 0$ for all $s < t < d_j$, then $y_2 - y_1 \leq h \leq 0$ on the interval (s,d_j). This implies that

$$\int_0^s (y_2-y_1-h) \geq \int_0^{d_j} (y_2-y_1-h)$$

$$= \int_0^{d_j} (y_2-y_1)$$

$$\geq 0.$$

Hence we have $y_2 \overset{y}{\geq} y_1 + h$. Since $y_2 - y_1 - h$ is a step function which takes at most $k-1$ nonzero values, it follows from the induction hypothesis that $y_2 - y_1 - h = \sum_{i=1}^{N} h_i'$ where $h_i' \in T$ for $i = 1, \ldots, N$, and $N \leq k-2$. This completes the proof. ‖

In Lemma 2.7, if we assume that y_1, y_2 are decreasing step functions, then the condition $y_2 \overset{y}{\geq} y_1$ is equivalent to $y_2 \overset{m}{\geq} y_1$. In addition, we can choose y_1+h_1, $y_1+h_1+h_2$, $\ldots$, $y_1+\sum_{i=1}^{N-1} h_i$ to be decreasing functions as shown in the following lemma.

2.8. Lemma. Let y_1, y_2 be decreasing step functions on $(0,1)$ such that $y_2 \overset{m}{\geq} y_1$. Then there exist $h_1, \ldots, h_N$ in T such that

i) $y_2 = y_1 + \sum\limits_{i=1}^{N} h_i$,

and

ii) $y_1 + h_1, \ldots, y_1 + \sum\limits_{i=1}^{N-1} h_i$ are decreasing functions.

Proof.

Define $h = a_1 I_{(c_1, c_1')} + a_j I_{(d_j', d_j)}$ as in the proof of Lemma 2.7. We need to show that $y_1 + h$ is decreasing. Note that

$$y_1(t) + h(t) = \begin{cases} y_2(t) & \text{if} \quad 0 < t < c_1', \\ y_1(t) & \text{if} \quad c_1' \le t \le d_j', \\ y_2(t) & \text{if} \quad d_j' < t \le d_j, \\ y_1(t) & \text{if} \quad d_j \le t < 1. \end{cases}$$

Since $a_1 > 0$, $y_1 + h$ is decreasing on a neighborhood of c_1'. Similarly, $a_j < 0$ implies that $y_1 + h$ is decreasing on a neighborhood of d_j'. Suppose that $d_j < 1$, then the choice of a_j indicates that for $\epsilon > 0$ sufficiently small, $y_2 - y_1 \ge 0$ on $(d_j, d_j + \epsilon)$. Since $y_1 + h = y_2$ on $(d_j', d_j]$, it follows that $y_1 + h$ is decreasing on the interval $(d_j', d_j + \epsilon)$. Thus $y_1 + h$ is decreasing on the interval $(0,1)$.

Note that $h \in T$ implies $y_1 + h \overset{y}{\ge} y_1$. Since $y_1 + h$, y_1 are decreasing functions, this is equivalent to $y_1 + h \overset{m}{\ge} y_1$. Following the same induction argument as in Lemma 2.7, we conclude that there exist $h_1, \ldots, h_N$ in T such that $y_2 = y_1 + h + \sum\limits_{i=1}^{N} h_i$ and that $y_1 + h + h_1, \ldots, y_1 + h + \sum\limits_{i=1}^{N-1} h_i$ are decreasing functions. This proves the lemma. $\|$

In the next theorem, we give a sufficient condition for a functional of $L_\infty(0,1)$ to be Schur-convex.

2.9.   Theorem. Let A be an invariant open convex subset of $L_\infty(0,1)$. Let ϕ be a continuous functional defined on A such that ϕ is constant over functions that are equivalent in distribution. If the Gâteaux differential $\frac{\partial\phi}{\partial h}(y) \geq 0$ for each $y \in \mathcal{D}_\infty \cap A$ and $h \in T$, then ϕ is Schur-convex on A.

Proof.

Since ϕ is constant over functions that are equivalent in distribution, it suffices to prove that ϕ is Schur-convex on $\mathcal{D}_\infty \cap A$.

Let y_1, $y_2 \in \mathcal{D}_\infty \cap A$ be right continuous and $y_2 \overset{m}{\geq} y_1$. Let $\epsilon > 0$ be arbitrary. Then for $i = 1$, 2, the sets $\{t: y_i(t^-) - y_i(t) > \epsilon\}$ are finite, where $y_i(t_0^-) = \underset{t<t_0}{\inf}\, y_i(t)$. Hence there exists a partition $0 < a_1 < \ldots < a_n < 1$ such that

$$y_i(a_k) - y_i(a_{k+1}^-) < \epsilon, \quad i = 1, 2; \quad k = 1, \ldots, n-1.$$

Define

$$
\begin{aligned}
y_{i\epsilon}(t) &= \frac{1}{a_1}\left[\int_0^{a_1} y_i(s)ds\right] I_{(0,a_1)}(t) \\
&\quad + \sum_{k=1}^{n-1} \frac{1}{a_{k+1} - a_k}\left[\int_{a_k}^{a_{k+1}} y_i(s)ds\right] I_{[a_k,a_{k+1})}(t) \\
&\quad + \frac{1}{1-a_n}\left[\int_{a_n}^{1} y_i(s)ds\right] I_{[a_n,1)}(t), \quad i = 1, 2.
\end{aligned}
$$

Then $y_{1\epsilon}$, $y_{2\epsilon}$ are decreasing step functions satisfying

$$\int_0^{a_k} y_{i\epsilon}(s)ds = \int_0^{a_k} y_i(s)ds \text{ for } k = 1, \ldots, n. \text{ This implies } y_{2\epsilon} \overset{m}{\geq} y_{1\epsilon}.$$

Since A is open and $\|y_i - y_{i\epsilon}\|_\infty < \epsilon$, for sufficiently small positive ϵ, $y_{1\epsilon}$, $y_{2\epsilon}$ are in A.

By Lemma 2.8, $y_{2\epsilon} - y_{1\epsilon} = \sum_{i=1}^{N} h_i$ for some $\{h_1, \ldots, h_N\} \subseteq T$, where $y_{1\epsilon} + h_1$, $\ldots$, $y_{1\epsilon} + \sum_{i=1}^{N-1} h_i$ are decreasing functions. The functions $y_{1\epsilon} + h_1$, $\ldots$, $y_{1\epsilon} + \sum_{i=1}^{N-1} h_i$ need not be elements of A. Since A is open, for sufficiently small positive θ, $y_{1\epsilon} + \theta h_1$, $\ldots$, $y_{1\epsilon} + \theta \sum_{i=1}^{N} h_i$ are decreasing functions in A satisfying

$$y_{1\epsilon} + \theta \sum_{i=1}^{N} h_i \overset{m}{\geq} y_{1\epsilon} + \theta \sum_{i=1}^{N-1} h_i \overset{m}{\geq} \ldots \overset{m}{\geq} y_{1\epsilon}. \text{ It now follows from Lemma 2.6}$$

that $\phi(y_{1\epsilon} + \theta \sum_{i=1}^{N} h_i) \geq \phi(y_{1\epsilon} + \theta \sum_{i=1}^{N-1} h_i) \geq \ldots \geq \phi(y_{1\epsilon})$. Next, we shall show that this implies $\phi(y_{2\epsilon}) \geq \phi(y_{1\epsilon})$. Note that we have just demonstrated that the set $\Theta = \{0 \leq \theta \leq 1 : \phi(y_{1\epsilon} + \theta \sum_{i=1}^{N} h_i) \geq \phi(y_{1\epsilon})\}$ is non-empty. Let $\theta_0 = \sup\{\theta : \theta \in \Theta\}$. Since ϕ is continuous, we have $\phi(y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i) \geq \phi(y_{1\epsilon})$, which shows that $\theta_0 \in \Theta$. Now suppose $\theta_0 < 1$. The preceding arguments show that for sufficiently small positive r, $y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i + r h_1$, $\ldots$, $y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i + r \sum_{i=1}^{N} h_i$ are decreasing functions in A and satisfy

$$\phi(y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i) \leq \phi(y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i + r h_1) \leq \ldots$$

$$\leq \phi(y_{1\epsilon} + \theta_0 \sum_{i=1}^{N} h_i + r \sum_{i=1}^{N} h_i). \text{ Thus } \theta_0 + r \in \Theta, \text{ which provides a}$$

contradiction to the assumption that $\theta_0 < 1$. We therefore conclude that $\theta_0 = 1$ and thus, $\phi(y_{2\epsilon}) \geq \phi(y_{1\epsilon})$.

Since the functional is continuous with respect to L_∞-norm, we conclude that $\phi(y_2) \geq \phi(y_1)$ by letting $\epsilon \to 0$. ∥

We now use this theorem to establish a sufficient condition for Schur-convex functional of $L_1(0,1)$.

2.10. Theorem. Let A be an invariant open convex subset of $L_1(0,1)$. Let ϕ be a continuous functional defined on A such that ϕ is constant over functions that are equivalent in distribution. If the Gâteaux differential $\frac{\partial \phi}{\partial h}(y) \geq 0$ for each $y \in \mathcal{D}_\infty \cap A$ and $h \in T$, then ϕ is Schur-convex on A.

Proof.

Since ϕ is constant over functions that are equivalent in distribution, it suffices to prove that ϕ is Schur-convex on $\mathcal{D}_1 \cap A$.

Let y_1, $y_2 \in \mathcal{D}_1 \cap A$ be right continuous and $y_2 \overset{m}{\geq} y_1$. Let $\epsilon > 0$ be arbitrary, then $\exists \delta > 0$ such that

$$\int_0^\delta |y_i(t)| dt < \frac{\epsilon}{4} \quad \text{and} \quad \int_{1-\delta}^1 |y_i(t)| dt < \frac{\epsilon}{4}, \quad i = 1, 2.$$

Since the y_i's are in $\mathcal{D}_1$, they are bounded on the interval $[\delta, 1-\delta]$. Define

$$y_{i\epsilon}(t) = \frac{1}{\delta}\left[\int_0^\delta y_i(s)ds\right]I_{(0,\delta)}(t) + y_i(t)I_{[\delta,1-\delta]}(t)$$

$$+ \frac{1}{\delta}\left[\int_{1-\delta}^1 y_i(s)ds\right]I_{(1-\delta,1)}(t), \quad i = 1, 2.$$

Then $y_{i\epsilon} \in \mathcal{D}_\infty$, $i = 1, 2$ and $y_{2\epsilon} \overset{m}{\geq} y_{1\epsilon}$. We also have

$$\|y_{i\epsilon}-y_i\|_1 = \int_0^\delta |y_{i\epsilon}-y_i| + \int_{1-\delta}^1 |y_{i\epsilon}-y_i| \leq \epsilon, \quad i = 1, 2.$$

Hence, for sufficiently small ϵ, $y_{i\epsilon} \in \mathcal{D}_\infty \cap A$. It now follows from Theorem 2.9 that $\phi(y_{2\epsilon}) \geq \phi(y_{1\epsilon})$. Since ϕ is a continuous functional, we obtain that $\phi(y_2) \geq \phi(y_1)$ by letting $\epsilon \to 0$. This completes the proof. $\|$

The following lemma is used to prove Theorem 2.12, which is an analogue of Theorem 2.10 for functionals on $L_1(0,1)$ which are non-decreasing with respect to the ordering of unrestricted majorization.

2.11 Lemma. Let y_1, $y_2 \in L_1(0,1)$ such that $y_2 \overset{u}{\geq} y_1$. For each $\epsilon > 0$, there exists a partition $0 < a_1 < \ldots < a_n < 1$ such that the step functions defined by

$$y_{i\epsilon}(t) = \frac{1}{a_1}\int_0^{a_1}\left[y_i(s)ds\right]I_{(0,a_1)}(t)$$

$$+ \sum_{k=1}^{n-1}\frac{1}{a_{k+1}-a_k}\left[\int_{a_k}^{a_{k+1}} y_i(s)ds\right]I_{[a_k,a_{k+1})}(t)$$

$$+ \frac{1}{1-a_n}\left[\int_{a_n}^1 y_n(s)ds\right]I_{[a_n,1)}(t), \quad i = 1, 2,$$

satisfy the following:

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(i) $\|y_{i\epsilon} - y_i\|_1 < \epsilon$, $i = 1, 2$

and

(ii) $y_{2\epsilon} \overset{u}{\geq} y_{1\epsilon}$.

Proof.

Note that if y_1, y_2 are continuous functions on the interval $[0,1]$, then (i) follows from the uniform continuity of y_1 and y_2. If y_i's are not continuous on $[0,1]$, we first approximate y_i's by continuous functions x_i on $[0,1]$ such that $\|y_i - x_i\|_1 < \frac{\epsilon}{3}$, $i = 1, 2$. Next, we find a partition $0 < a_1 < \ldots < a_n < 1$ such that the step functions defined by

$$x_{i\epsilon}(t) = \frac{1}{a_1}\left[\int_0^{a_1} x_i(s)ds\right]I_{(0,a_1)}(t)$$

$$+ \sum_{k=1}^{n-1} \frac{1}{a_{k+1}-a_k}\left[\int_{a_k}^{a_{k+1}} x_i(s)ds\right]I_{[a_k,a_{k+1})}(t)$$

$$+ \frac{1}{1-a_n}\left[\int_{a_n}^{1} x_i(s)ds\right]I_{[a_n,1)}(t), \quad i = 1, 2$$

satisfy $\|x_{i\epsilon} - x_i\|_1 < \frac{\epsilon}{3}$.

Now, define the step functions $y_{1\epsilon}$, $y_{2\epsilon}$ by

$$y_{i\epsilon}(t) = \frac{1}{a_1}\left[\int_0^{a_1} y_i(s)ds\right]I_{(0,a_1)}(t)$$

$$+ \sum_{k=1}^{n-1} \frac{1}{a_{k+1}-a_k}\left[\int_{a_k}^{a_{k+1}} y_i(s)ds\right]I_{[a_k,a_{k+1})}(t)$$

$$+ \frac{1}{1-a_n}\left[\int_{a_n}^{1} y_i(s)ds\right]I_{[a_n,1)}(t), \quad i = 1, 2.$$

Then

$$\|y_{i\epsilon} - x_{i\epsilon}\|_1 = \left| \int_0^{a_1} [y_i(s) - x_i(s)]ds \right|$$

$$+ \sum_{k=1}^{n-1} \left| \int_{a_k}^{a_{k+1}} [y_i(s) - x_i(s)]ds \right|$$

$$+ \left| \int_{a_n}^1 [y_i(s) - x_i(s)]ds \right|$$

$$\leq \|x_i - y_i\|_1, \quad i = 1, 2.$$

Thus

$$\|y_i - y_{i\epsilon}\|_1 \leq \|y_i - x_i\|_1 + \|x_i - x_{i\epsilon}\|_1 + \|x_{i\epsilon} - y_{i\epsilon}\|_1$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad i = 1, 2.$$

This proves the first part of the lemma.

Let $y_2 \overset{u}{\geq} y_1$. Then for any partition $0 < a_1 < \ldots < a_n < 1$, the step functions $y_{i\epsilon}$, $y_{2\epsilon}$ satisfy that $y_{2\epsilon} \overset{u}{\geq} y_{1\epsilon}$. This proves (ii). ∥

2.12 Theorem. Let A be an open convex subset of $L_1(0,1)$ and let ϕ be a continuous functional on A such that the Gâteaux differentials $\frac{\partial \phi}{\partial h}(y) \geq 0$ for $y \in A$ and $h \in T$. Then y_1, $y_2 \in A$ and $y_2 \overset{u}{\geq} y_1$ imply that $\phi(y_2) \geq \phi(y_1)$.

Proof.

We shall first prove the theorem for step functions. Let y_1, y_2 be step functions in A and $y_2 \overset{u}{\geq} y_1$. Then by Lemma 2.7, $y_2 - y_1 = \sum_{i=1}^N h_i$ for some $\{h_1, \ldots, h_N\} \subseteq T$. Since A is open, for

sufficiently small positive θ, $y_1 + \theta h_1$, $\ldots$, $y_1 + \theta \sum_{i=1}^{N} h_i$ are functions in A and satisfy that

$$y_1 + \theta \sum_{i=1}^{N} h_i \overset{y}{\geq} y_1 + \theta \sum_{i=1}^{N-1} h_i \overset{y}{\geq} \ldots \geq y_1 + \theta h_1 \overset{y}{\geq} y_1.$$

It now follows from Lemma 2.6 that
$\phi(y_1 + \theta \sum_{i=1}^{N} h_i) \geq \ldots \geq \phi(y_1 + \theta h_1) \geq \phi(y_1)$. Define
$\Theta = \{0 \leq \theta \leq 1: \phi(y_1 + \theta \sum_{i=1}^{N} h_i) \geq \phi(y_i)\}$ and $\theta_0 = \sup\{\theta: \theta \in \Theta\}$.
Following the same argument as in the proof of Theorem 2.9, we
can show that $\theta_0 = 1$. Hence,

$$\phi(y_2) = \phi(y_1 + \sum_{i=1}^{N} h_i) \geq \phi(y_1).$$

In general, let y_1, $y_2 \in L_1(0,1)$ and $y_2 \overset{y}{\geq} y_1$. Let $\epsilon > 0$.
By Lemma 2.11 there exist step functions $y_{1\epsilon}$, $y_{2\epsilon}$ such that
$\|y_{i\epsilon} - y_i\|_1 < \epsilon$ for $i = 1, 2$ and $y_{2\epsilon} \overset{y}{\geq} y_{1\epsilon}$. Since A is open, for
sufficiently small ϵ, $y_{1\epsilon}$, $y_{2\epsilon}$ are functions in A. Thus
$\phi(y_{2\epsilon}) \geq \phi(y_{1\epsilon})$. Since ϕ is continuous, we conclude that
$\phi(y_2) \geq \phi(y_1)$ by letting $\epsilon \to 0$. $\|$

3. Applications.

The inequality given in Theorem 1.4 can be reformulated as the
statement that the functional defined by

$$\phi(x) = \int_0^1 \log[\int_0^1 u(t)^{x(s)} dt] ds$$

is Schur-convex. By Theorem 2.10, this is equivalent to the condition
$\frac{\partial \phi}{\partial h}(x) \geq 0 \; \forall \; x \in \mathcal{D}_\infty$, $\forall \; h \in T$. This condition can be verified as follows.

Using Holder's inequality, we note that the function

$M(\alpha) = \log\|u\|_{\alpha}^{\alpha}$ is convex in α, and thus

$$M'(x) = \int_0^1 u(t)^{\alpha} \log u(t)dt / \int_0^1 u(t)^{\alpha}dt$$

is increasing in α. Let $x \in \mathcal{D}_{\infty}$ and $h \in T$, then both x and h are functions decreasing on their supports, and $\int_0^1 h(t)dt = 0$. This implies that

$$\frac{\partial\phi}{\partial h}(x) = \int_0^1 \left[\frac{\int_0^1 u(t)^{x(s)} \log u(t)dt}{\int_0^1 u(t)^{x(s)}dt}\right]h(s)ds \geq 0.$$

More generally, we can replace the function $u(t)^{x(s)}$ by functions of the form $\psi(t,z)$ which are log convex in z for fixed t. This is the result of Proschan and Sethuraman (1976), which we will state below.

3.1. Theorem. Let the function $\psi(t,z)$ on $(0,1) \times (-\infty,\infty)$ be a log convex function in z for fixed t, and the partial derivative $\psi_2(t,z) = \frac{\partial}{\partial z}\psi(t,z)$ exists. Also let $\sup\limits_{|z| \leq k} \psi(t,z)$ belong to $L_1(0,1)$ for each $k < \infty$. For any bounded measurable function x on $(0,1)$, define

$$M_{\psi}(x) = \int_0^1 \log\left[\int_0^1 \psi(t,x(s))dt\right]ds.$$

Then M_{ψ} is Schur-convex.

Proof.

If follows from Artin's Theorem (1931) that the positive linear combination $\int_0^1 \psi(t,z)dt$ is log convex in z. Let $x \in \mathcal{D}_\infty$ and $h \in T$, then

$$\frac{\partial M_\psi}{\partial h}(x) = \int_0^1 \left[\frac{\int_0^1 \psi_2(t,x(s))dt}{\int_0^1 \psi(t,x(s))dt} \right] h(s)ds \geq 0,$$

which implies that M_ψ is Schur-convex. ||

Next, we shall study an application of unrestricted majorization to peakedness ordering of symmetric distributions.

Let X and Y be random variables possessing densities symmetric about zero. According to the definition of peakedness given by Birnbaum (1948), X is more peaked than Y, ($X \overset{P}{\geq} Y$ in symbols), if $P(X \leq t) \geq P(Y \leq t)$ for all $t \geq 0$. Let f and g be the densities of X and Y respectively. Then the condition $X \overset{P}{\geq} Y$ is equivalent to $f \overset{U}{>} g$ on the interval $(0,\infty)$.

Birnbaum (1948) showed that under appropriate conditions, $X_1 \overset{P}{\geq} Y_1$ and $X_2 \overset{P}{\geq} Y_2$ imply that $X_1 + X_2 \overset{P}{\geq} Y_1 + Y_2$. This result can be obtained by considering certain order preserving functionals. We first introduce some simplifying notations.

For s > 0, define

$$\chi_s(x) = I(|x| < s)$$

and, for a symmetric function h, define

$$h(s,x) = (h * \chi_s)(x)$$

$$= \int h(x-y)\chi_s(y)dy,$$

and

$$h(s,0) = \int h(-y)\chi_s(y)dy$$

$$= \int h(y)\chi_s(y)dy.$$

Note that

$$h(s,x) = \begin{cases} \frac{1}{2}[h(x+s,0) - h(x-s,0)] & \text{is } s < x, \\ \frac{1}{2}[h(x+s,0) + h(-x+s,0)] & \text{if } -s \le x \le s, \\ \frac{1}{2}[h(-x+s,0) - h(-x-s,0)] & \text{if } x < -s. \end{cases}$$

We need the following lemma.

3.2. Lemma. Let $C = \{h: h \text{ symmetric and } h(s,0) \ge 0 \text{ for all } s > 0\}$. Let g be symmetric and decreasing on $(0,\infty)$. Then $h * g \in C$ for all $h \in C$, i.e, $(h*g*\chi_s)(0) \ge 0$ for $h \in C$ and $s > 0$.

Proof.

Let $h \in C$ and $s > 0$. Then

$$(h*g*\chi_s)(0) = \int (h*\chi_s)(x)g(-x)dx$$

$$= \int h(s,x)g(-x)dx$$

$$= \frac{1}{2}\left\{ \int_{x>s} [h(x+s,0) - h(x-s,0)]g(x)dx \right.$$

$$+ \int_{-s \le x \le s} [h(x+s,0) - h(-x+s,0)]g(x)dx$$

$$+ \left. \int_{x<-s} [h(-x+s,0) - h(-x-s,0)]g(x)dx \right\}$$

$$= \frac{1}{2}\left[\int_{x>-s} h(x+s,0)g(x)dx + \int_{x<s} h(-x+s,0)g(x)dx \right.$$

$$\left. - \int_{x<-s} h(-x-s,0)g(x)dx - \int_{x>s} h(x-s,0)g(x)dx \right].$$

Let $y = x + s$ in the first integral, $y = -x + s$ in the second integral, $y = -x - s$ in the third integral and $y = x - s$ in the fourth integral. We get

$$(h*g*\chi_s)(0) = \frac{1}{2}\left[\int_{y>0} h(y,0)g(y-s)dy\right.$$

$$+ \int_{y>0} h(y,0)g(-y+s)dy - \int_{y>0} h(y,0)g(-y-s)dy$$

$$\left. - \int_{y>0} h(y,0)g(y+s)dy\right].$$

By the symmetry of g,

$$(h*g*\chi_s)(0) = \int_{y>0} h(y,0)[g(y-s) - g(y+s)]dy.$$

Since $h(y,0) \geq 0$ for all $y > 0$, and $g(y-s) - g(y+s) \geq 0$ for $y > 0$ and $s > 0$, we conclude that $(h*g*\chi_s)(0) \geq 0$. ‖

We may now obtain the following result.

3.3. Theorem. Let X_1, X_2, Y_1, Y_2 be independent symmetric random variables on $(-1,1)$ with densities f_1, f_2, g_1, g_2, respectively. Let f_1, g_2 be non-increasing on $(0,1)$. Let $X_i \overset{P}{\geq} Y_i$, for $i = 1, 2$. Then $X_1 + X_2 \overset{P}{\geq} Y_1 + Y_2$.

Proof.

We will first establish that $X_1 + X_2 \overset{P}{\geq} X_1 + Y_2$.

Fix f_1. For each $f \in L_1(0,1)$, let f_S be the symmetric function on $(-1,1)$ defined by

$$f_S(t) = f(|t|).$$

For each $s > 0$, define a functional on $L_1(0,1)$ by

$$\phi_s(f) = \iint I(|x_1 + x_2| \leq s) f_1(x_1) f_S(x_2) dx_1 dx_2.$$

Let $T(0,1)$ be the class of nonnegative functions u on $(0,1)$ with $\int_0^1 u(t) dt = \frac{1}{2}$. Note that for $f \in T(0,1)$,

$$\phi_s(f) = P(|X_1 + Z| \leq s),$$

where X_1, Z are independent random variables with densities f_1, f_S respectively.

We shall show that for each $s > 0$, ϕ_s is non-decreasing with respect to the ordering of unrestricted majorization on $T(0,1)$. Let $s > 0$. Let $f \in T(0,1)$ and $h \in T$. Then,

$$
\begin{aligned}
\frac{\partial \phi_s}{\partial h}(f) &= \lim_{\theta \to 0} \frac{1}{\theta} \iint I(|x_1 + x_2| \leq s) \Big\{ f_1(x_1) [f_S(x_2) + \theta h_S(x_2)] \\
&\qquad\qquad - f_1(x_1) f_S(x_2) \Big\} dx_1 dx_2 \\
&= \iint I(|x_1 + x_2| \leq s) f_1(x_1) h_S(x_2) dx_1 dx_2 \\
&= (f_1 * h_S * \chi_S)(0).
\end{aligned}
$$

Since $h \in T$, $h_S(t,0) \geq 0$ for all $t > 0$. By Lemma 3.2, $(f_1 * h_S * \chi_S)(0) \geq 0$. It now follows from Theorem 2.12 that ϕ_s is nondecreasing with respect to the ordering of unrestricted majorization on $T(0,1)$.

Note that $X_2 \overset{P}{\geq} Y_2$ implies that $f_2 \overset{y}{\geq} g_2$ when these are considered as elements of $T(0,1)$. We now have

$$
\begin{aligned}
P(|X_1 + X_2| \leq s) &= \phi_s(f_2) \\
&\geq \phi_s(g_2) \\
&= P(|X_1 + Y_2| \leq s), \quad \text{for all } s > 0.
\end{aligned}
$$

Thus $X_1 + X_2 \overset{P}{\geq} X_1 + Y_2$. Similarly, we can establish that $X_1 + Y_2 \overset{P}{\geq} Y_1 + Y_2$. Hence $X_1 + X_2 \overset{P}{\geq} Y_1 + Y_2$. $\|$

REFERENCES

[1] Artin, E. (1931). Einfürhring in die Theorie der Gammafunktion. Hambuger Mathematische Einzelschriften 11.

[2] Birnbaum, Z.W. (1948). On random variables with comparable peakedness. Ann. Math. Stat. 19, 76 - 81.

[3] Chong, K.M. (1976). Doubly stochastic operators and rearrangement theorems. J. Math. Anal. Appl. 56, 309 - 316.

[4] Day, P.W. (1973). Decreasing rearrangements and doubly stochastic operators. Trans. Amer. Math. Soc. 178, 383-392.

[5] Hardy, G. H., Littlewood, J.E., and Polya, G. (1934). Inequalities. Cambridge University Press.

[6] Marshall, A.W., and Olkin, I. (1979). Inequalities: Theory of Majorization and Its Applications. Academic Press.

[7] Ostrowski, A.M. (1952). Sur quelques applications des fonctions convexes et concaves au sens de I. Schur. J. Math. Pures Appl. 31, 253 - 292.

[8] Proschan, F., and Sethuraman, J. (1977). Two generalizations of Muirhead's Theorem. Bull. Cal. Math. Soc. 69, 341 - 344.

[9] Ross, S.M. (1982). Stochastic Processes. Wiley, New York.

[10] Ryff, J.V. (1963). On the representation of doubly stochastic operators. Pacific J. Math. 13, 1379 - 1386.

[11] Ryff, J.V. (1967). On Muirhead's Theorem. Pacific J. Math. 21, 567 - 576.

[12] Ryff, J.V. (1970). Measure preserving transformations and rearrangements. J. Math. and Anal. Appl. 31, 449 - 458.

[13] Schur, I. (1923). Über eine Klasse von Mittelbildunger mit Anwendungen die Determinanten - Theorie Sitzungsber. Berlin Math. Gesellschaft 22, 9 - 20.

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